Quantum Random Access Codes with Shared Randomness

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Outline

Introduction

Random access codes (RACs)

$n \overset{p}{\mapsto} m$ random access code

1. Alice encodes n bits into m and sends them to Bob ($n > m$).
2. Bob must be able to restore any one of the n initial bits with probability $\geq p$.

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   - *classical* RAC: Alice encodes n classical bits into 1 classical bit,
   - *quantum* RAC: Alice encodes n classical bits into 1 qubit.

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   - *classical* RAC: Alice encodes n classical bits into 1 classical bit,
   - *quantum* RAC: Alice encodes n classical bits into 1 qubit.

In quantum case the state collapses after recovery of one bit, so we may loose the other bits.

Classical random access codes with shared randomness

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Complexity measures

We are interested in the worst case success probability of RAC. However, it is simpler to consider the average case success probability. In the next few slides we will see that there is a way how to switch between these two.

Different kinds of classical RACs

Definition

A *pure classical* $n \mapsto 1$ *RAC* is an ordered tuple $(E, D_1, \ldots, D_n)$ that consists of *encoding function* $E : \{0,1\}^n \mapsto \{0,1\}$ and n *decoding functions* $D_i : \{0,1\} \mapsto \{0,1\}$.

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Definition

A *mixed classical* $n \mapsto 1$ *RAC* is an ordered tuple $(P_E, P_{D_1}, \ldots, P_{D_n})$ of probability distributions. P_E is a distribution over encoding functions and P_{D_i} over decoding functions.

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Definition

A *classical* $n \mapsto 1$ *RAC with shared randomness* (SR) is a probability distribution over pure classical RACs.

Playing with randomness

Yao's principle

$$\boxed{\min_{\mu} \max_{\mathcal{D}} \Pr_{\mu}[\mathcal{D}(x) = f(x)] = \max_{\mathcal{A}} \min_{x} \Pr[\mathcal{A}(x) = f(x)]}$$

The following notations are used:

- f - some function we want to compute,
- $\Pr_{\mu}[\mathcal{D}(x) = f(x)]$ – success probability of deterministic algorithm $\mathcal{D}$ with input x distributed according to μ,
- $\Pr[\mathcal{A}(x) = f(x)]$ – success probability of probabilistic algorithm $\mathcal{A}$ on input x.

Obtaining upper and lower bounds

Upper bound

We can take any input distribution μ_0 that seems to be "hard" for deterministic algorithms and find p such that

$$\max_{\mathcal{D}} \Pr_{\mu_0}[\mathcal{D}(x) = f(x)] \leq p$$

Then according to Yao's principle the worst case success probability of the best probabilistic algorithm is upper bounded by p.

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Lower bound

Any pure RAC with average case success probability p can be turned into a RAC with shared randomness having worst case success probability p by jointly randomizing the input (requires $n + \log n$ shared random bits). Thus we can obtain a lower bound by randomizing any pure RAC.

The “hardest” input distribution

Matching upper and lower bounds

The lower bound was obtained by simulating uniform input distribution. Since any input distribution μ_0 can be used for the upper bound, we can use the uniform distribution as well – then both bounds will match. Hence for pure random access codes uniform input distribution is the “hardest”.

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Conclusion

Best pure RAC for uniformly distributed input (average success prob.) $\overset{\text{input}}{\underset{\text{randomization}}{\Longrightarrow}}$ Best RAC with SR (worst case success prob.)

Optimal classical RAC

Optimal decoding

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Optimal encoding

Once Alice knows that Bob's decoding function is $D(x) = x$, she simply encodes the majority of all bits.

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Counting

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Using Stirling's approximation:

$$p(n) \approx \frac{1}{2} + \frac{1}{\sqrt{2\pi n}}$$

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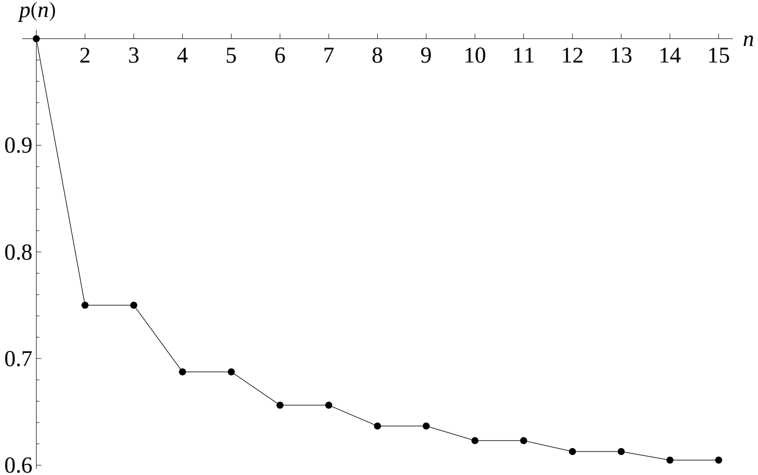

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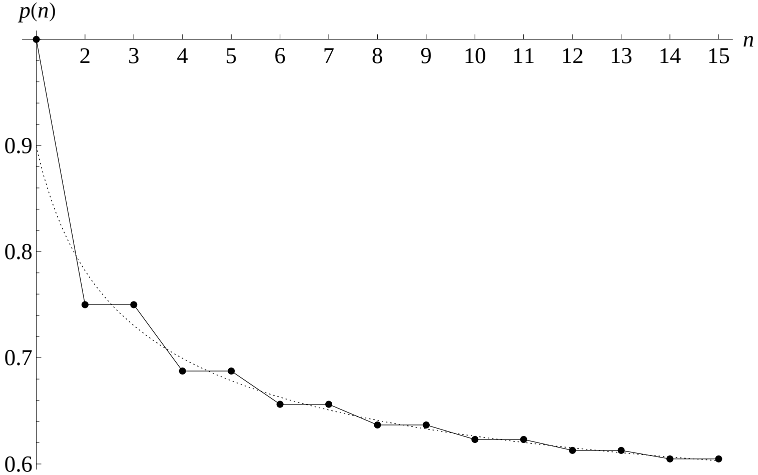
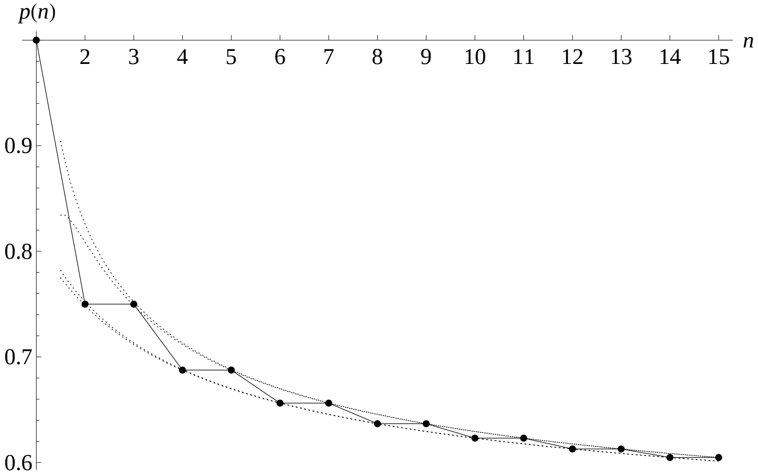


Probability of success

Using inequalities $\sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n+1}} < n! < \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\frac{1}{12n}}$



Quantum random access codes with shared randomness

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Alice encodes a classical bit string into a qubit state and sends it to Bob. We will use the Bloch sphere to visualize these states.

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Bloch vector

$$|\psi\rangle = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} \end{pmatrix}$$

$0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$

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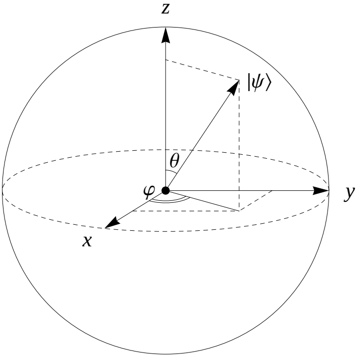

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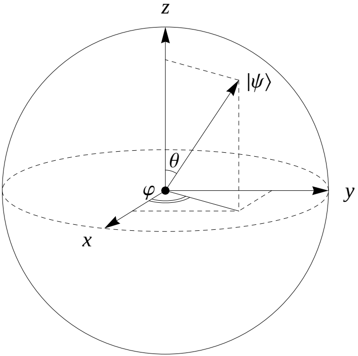

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$$\Updownarrow$$

$$\vec{r} = (x, y, z)$$

$$\begin{cases} x = \sin\theta\cos\varphi \\ y = \sin\theta\sin\varphi \\ z = \cos\theta \end{cases}$$

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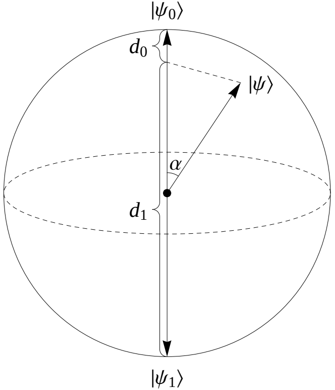
Measurement



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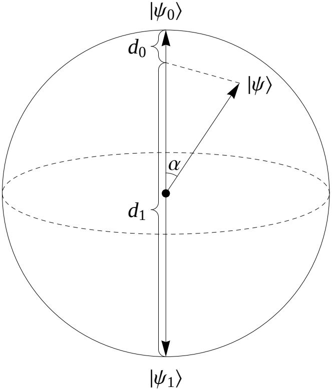

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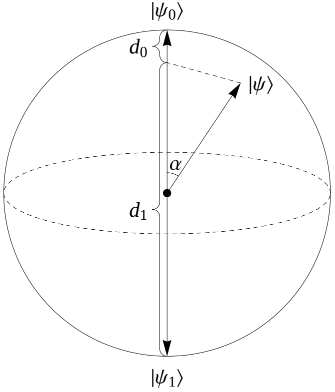

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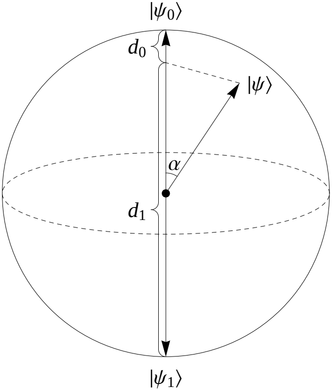

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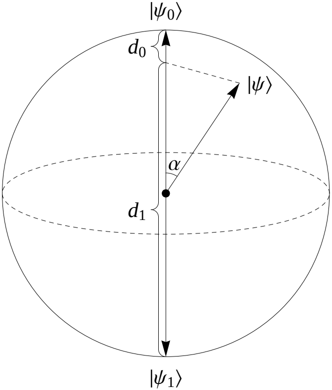

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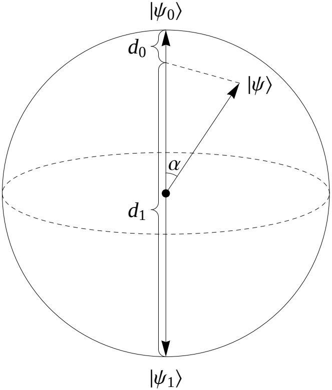

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Known results

Pure strategies

Only two specific QRACs are known when *pure quantum strategies* are allowed. That means:

1. Alice prepares a pure state,
2. Bob uses a projective measurement (not a POVM),
3. the worst case success probability must be at least $\frac{1}{2}$.

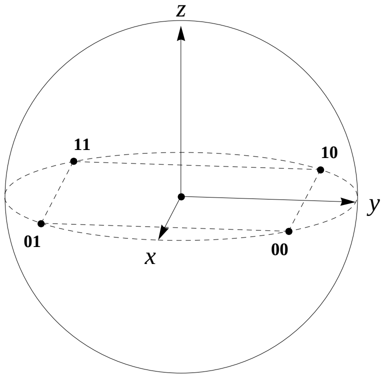
Note: shared randomness is not allowed.

Known QRACs

$2 \overset{p}{\mapsto} 1$ code

There is a $2 \overset{p}{\mapsto} 1$ code with $p = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.85$.

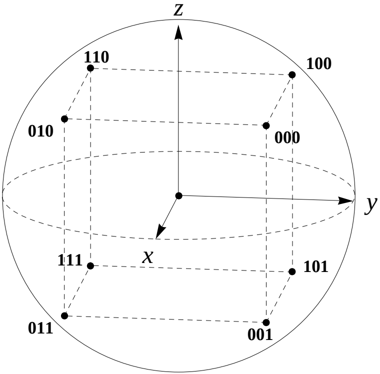
This code is optimal. [quant-ph/9804043]



Known QRACs

$3 \overset{p}{\mapsto} 1$ code

There is a $3 \overset{p}{\mapsto} 1$ code with $p = \frac{1}{2} + \frac{1}{2\sqrt{3}} \approx 0.79$.
This code is optimal. [I.L. Chuang]



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There is no $4 \overset{p}{\mapsto} 1$ code for $p > \frac{1}{2}$.

Proof idea – it is not possible to cut the surface of the Bloch sphere into 16 parts with 4 planes passing through its center. [quant-ph/0604061]

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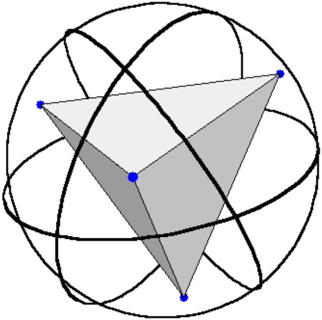

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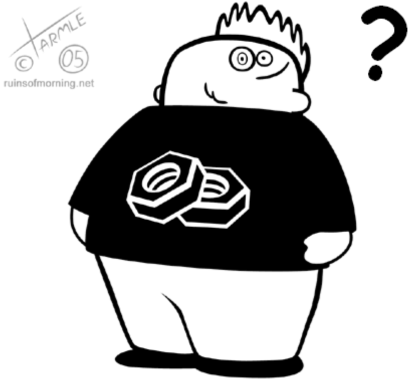

What can we do about this?

Use shared randomness!

Different kinds of quantum RACs

Definition

Pure quantum $n \mapsto 1$ *RAC* is an ordered tuple $(E, M_1, \ldots, M_n)$ that consists of *encoding function* $E : \{0,1\}^n \mapsto \mathbb{C}^2$ and n orthogonal measurements: $M_i = \left\{ \left| \psi_0^i \right\rangle, \left| \psi_1^i \right\rangle \right\}$.

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Mixed quantum $n \mapsto 1$ *RAC* is an ordered tuple $(P_E, P_{M_1}, \ldots, P_{M_n})$ of probability distributions. P_E is a distribution over encoding functions E and P_{M_i} are probability distributions over orthogonal measurements of qubit.

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Definition

Quantum $n \mapsto 1$ *RAC with shared randomness* is a probability distribution over pure quantum RACs.

Finding QRACs with SR

Recall

Let $\vec{r}_1$ and $\vec{r}_2$ be the Bloch vectors corresponding to qubit states $|\psi_1\rangle$ and $|\psi_2\rangle$. Then $|\langle\psi_1|\psi_2\rangle|^2 = \frac{1}{2}(1 + \vec{r}_1 \cdot \vec{r}_2)$.

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Optimize

The average success probability is:

$$p(\{\vec{v}_i\}, \{\vec{r}_x\}) = \frac{1}{2^n \cdot n} \sum_{x \in \{0,1\}^n} \sum_{i=1}^{n} \frac{1 + (-1)^{x_i} \vec{v}_i \cdot \vec{r}_x}{2}$$

Optimal quantum encoding

Probability

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Observe

$$\max_{\{\vec{v}_i\},\{\vec{r}_x\}} \sum_{x \in \{0,1\}^n} \left(\vec{r}_x \cdot \sum_{i=1}^{n} (-1)^{x_i} \vec{v}_i \right) = \max_{\{\vec{v}_i\}} \sum_{x \in \{0,1\}^n} \left\| \sum_{i=1}^{n} (-1)^{x_i} \vec{v}_i \right\|$$

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Optimal encoding

Given $\{\vec{v}_i\}$, optimal encoding of x is a unit vector $\vec{r}_x$ in direction of

$$\sum_{i=1}^{n} (-1)^{x_i} \vec{v}_i$$

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$$\max_{\{\vec{v}_i\},\{\vec{r}_x\}} \sum_{x \in \{0,1\}^n} \left(\vec{r}_x \cdot \sum_{i=1}^{n} (-1)^{x_i} \vec{v}_i \right) = \max_{\{\vec{v}_i\}} \sum_{x \in \{0,1\}^n} \left\| \sum_{i=1}^{n} (-1)^{x_i} \vec{v}_i \right\|$$

Optimal encoding

Given $\{\vec{v}_i\}$, optimal encoding of x is a unit vector $\vec{r}_x$ in direction of

$$\sum_{i=1}^{n} (-1)^{x_i} \vec{v}_i$$

Note: if all $\vec{v}_i$ are equal, this corresponds to the optimal classical (majority) encoding.

Upper bound

Success probability using optimal encoding

$$p(\{\vec{v}_i\}) = \frac{1}{2}\left(1 + \frac{1}{2^n \cdot n} \sum_{a \in \{1,-1\}^n} \left\| \sum_{i=1}^{n} a_i \vec{v}_i \right\| \right)$$

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Lemma

For any unit vectors $\vec{v}_1, \ldots, \vec{v}_n$ *we have:*

$$\sum_{a_1,\ldots,a_n \in \{1,-1\}} \|a_1 \vec{v}_1 + \cdots + a_n \vec{v}_n\|^2 = n \cdot 2^n$$

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Think of this as a generalization of the parallelogram identity

$$\|\vec{v}_1 + \vec{v}_2\|^2 + \|\vec{v}_1 - \vec{v}_2\|^2 = 2\left(\|\vec{v}_1\|^2 + \|\vec{v}_2\|^2\right)$$

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To remove the square, use inequality that follows form $(x-y)^2 \geq 0$:

$$xy \leq \frac{1}{2}(x^2 + y^2)$$

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Theorem

For any $n \overset{p}{\mapsto} 1$ *QRAC with shared randomness:* $p \leq \dfrac{1}{2} + \dfrac{1}{2\sqrt{n}}$.

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For any $n \overset{p}{\mapsto} 1$ *QRAC with shared randomness:* $p \leq \dfrac{1}{2} + \dfrac{1}{2\sqrt{n}}$.

Note: this holds even if Bob can use a POVM measurement.

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Success probability using optimal encoding

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Random measurements

Alice and Bob can sample each $\vec{v}_i$ at random. This can be done near uniformly given enough shared randomness. Observe

$$\mathop{\mathbb{E}}_{\{\vec{v}_i\}} \left(\sum_{a \in \{1,-1\}^n} \left\| \sum_{i=1}^{n} a_i \vec{v}_i \right\| \right) = 2^n \cdot \mathop{\mathbb{E}}_{\{\vec{v}_i\}} \left\| \sum_{i=1}^{n} \vec{v}_i \right\|$$

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What is the average distance traveled in 3D after n steps of unit length if the direction of each step is chosen uniformly at random?

Lower bound

Success probability using random measurements

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Random walk

Probability density to arrive at point $\vec{R}$ after performing $n \gg 1$ steps of random walk [Chandrasekhar 1943]:

$$W(\vec{R}) = \left(\frac{3}{2\pi n}\right)^{3/2} e^{-3\left\|\vec{R}\right\|^2/2n}$$

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Thus the average distance traveled is

$$\int_0^\infty 4\pi R^2 \cdot R \cdot W(R) \cdot dR = 2\sqrt{\frac{2n}{3\pi}}$$

Lower bound

Success probability using random measurements

$$\underset{\{\vec{v}_i\}}{\mathbb{E}}\, p(\{\vec{v}_i\}) = \frac{1}{2} + \sqrt{\frac{2}{3\pi n}}$$

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Success probability using random measurements

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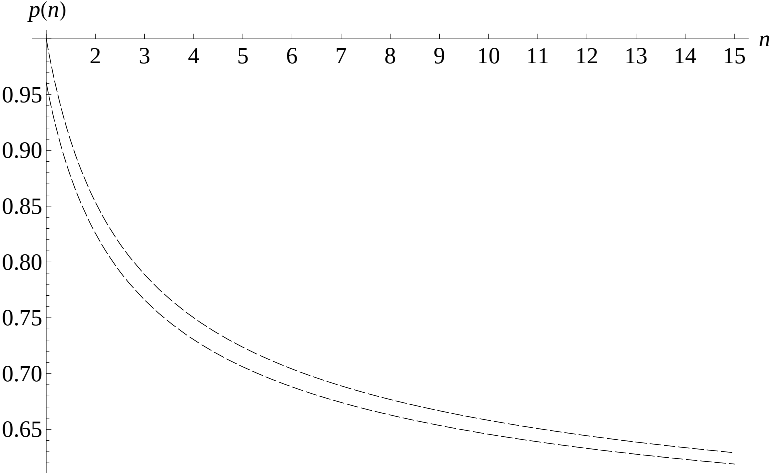
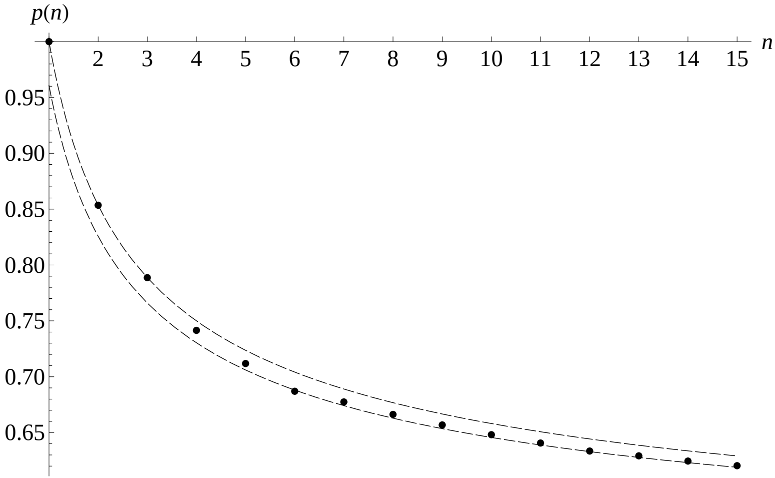
Theorem

There exists $n \overset{p}{\mapsto} 1$ QRAC with SR such that $p = \dfrac{1}{2} + \sqrt{\dfrac{2}{3\pi n}}$.

Quantum upper and lower bounds



Quantum upper and lower bounds



Black dots correspond to a lower bound obtained using measurements on orthogonal Bloch vectors.

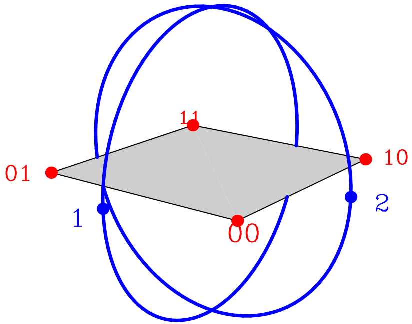
Results of numerical optimization

Results of numerical optimization

See our homepage

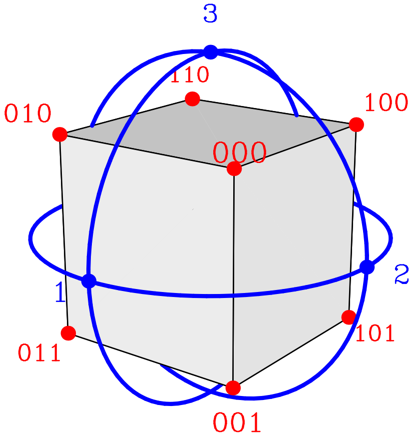
`http://home.lanet.lv/~sd20008/RAC/RACs.htm`

Numerical $2 \mapsto 1$ QRAC



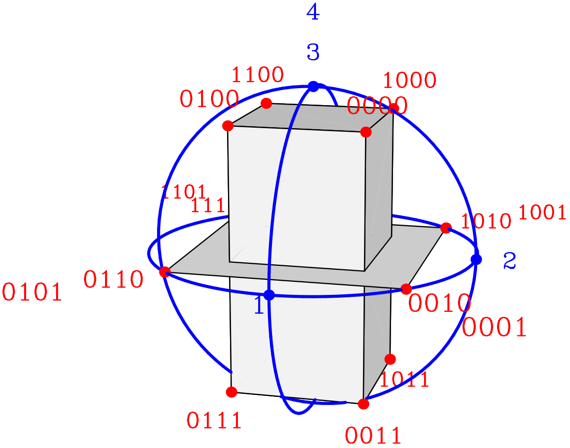
$$p = \frac{1}{2} + \frac{1}{2\sqrt{2}} \approx 0.8535533906$$

Numerical $3 \mapsto 1$ QRAC



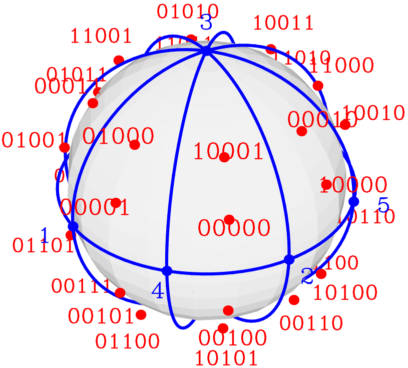
$$p = \frac{1}{2} + \frac{1}{2\sqrt{3}} \approx 0.7886751346$$

Numerical $4 \mapsto 1$ QRAC



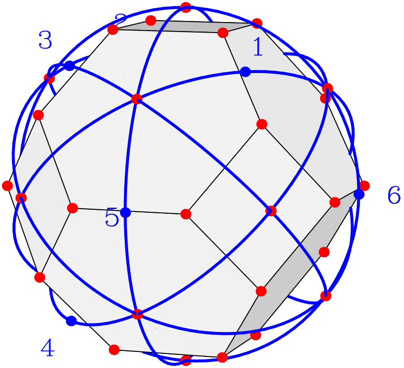
$$p = \frac{1}{2} + \frac{1+\sqrt{3}}{8\sqrt{2}} \approx 0.7414814566$$

Numerical $5 \mapsto 1$ QRAC



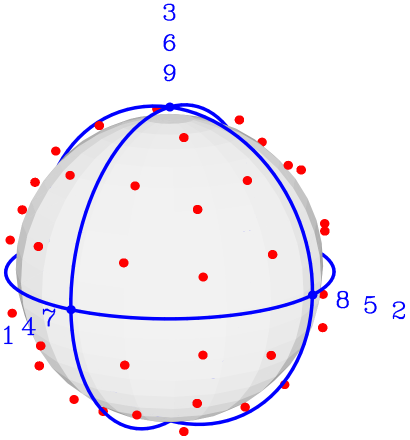
$$p = \frac{1}{2} + \frac{1}{20}\sqrt{2(5+\sqrt{17})} \approx 0.7135779205$$

Numerical $6 \mapsto 1$ QRAC



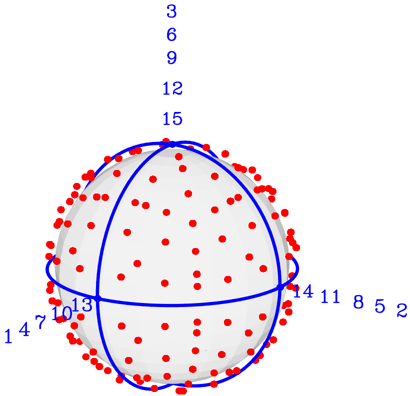
$$p = \frac{1}{2} + \frac{2 + \sqrt{3} + \sqrt{15}}{16\sqrt{6}} \approx 0.6940463870$$

Numerical $9 \mapsto 1$ QRAC



$$p = \frac{1}{2} + \frac{192 + 10\sqrt{3} + 9\sqrt{11} + 3\sqrt{19}}{384} \approx 0.6568927813$$

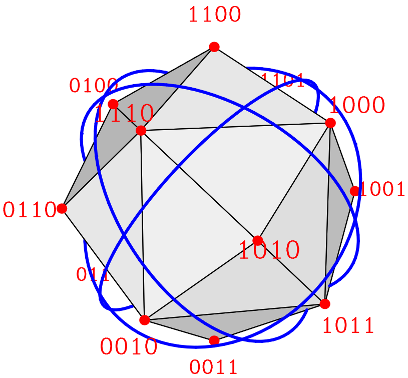
Numerical $15 \mapsto 1$ QRAC



$$p = \frac{1}{2} + \frac{152\sqrt{3}+100\sqrt{11}+50\sqrt{19}+20\sqrt{35}+5\sqrt{43}+2\sqrt{51}+\sqrt{59}}{8192} \approx 0.6203554614$$

Symmetric (but not optimal) constructions

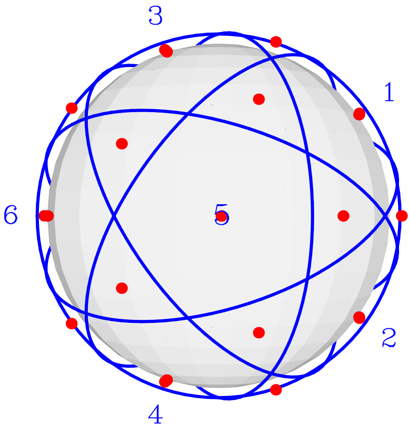
Symmetric $4 \mapsto 1$ QRAC



$$p = \frac{1}{2} + \frac{2+\sqrt{3}}{16} \approx 0.7332531755$$

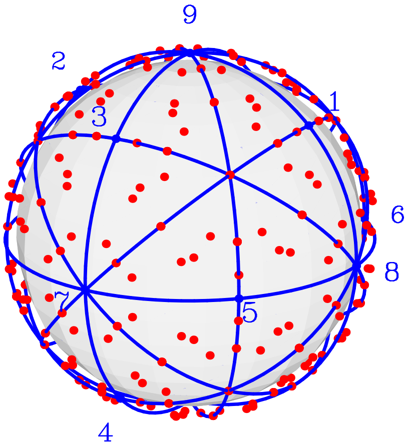
$$\leq 0.7414814566$$

Symmetric $6 \mapsto 1$ QRAC



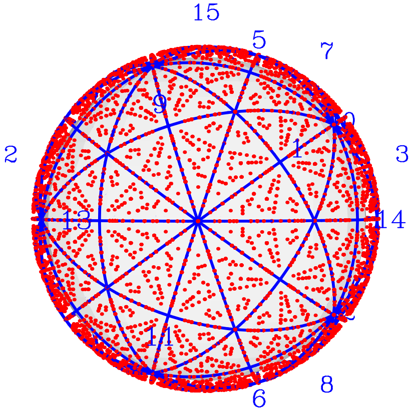
$$p = \frac{1}{2} + \frac{\sqrt{5}}{32} + \frac{1}{96}\sqrt{75 + 30\sqrt{5}} \approx 0.6940418856$$
$$\leq 0.6940463870$$

Symmetric $9 \mapsto 1$ QRAC



$$p \approx 0.6563927998$$
$$\leq 0.6568927813$$

Symmetric $15 \mapsto 1$ QRAC



$$p \approx 0.6201829084$$
$$\leq 0.6203554614$$

Summary

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Classical RACs with SR

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- exact success probability of optimal RAC:
  $p(2m) = p(2m+1) = \dfrac{1}{2} + \dfrac{1}{2^{2m+1}}\binom{2m}{m},$

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- upper bound: $p(n) \leq \dfrac{1}{2} + \dfrac{1}{2\sqrt{n}}$,

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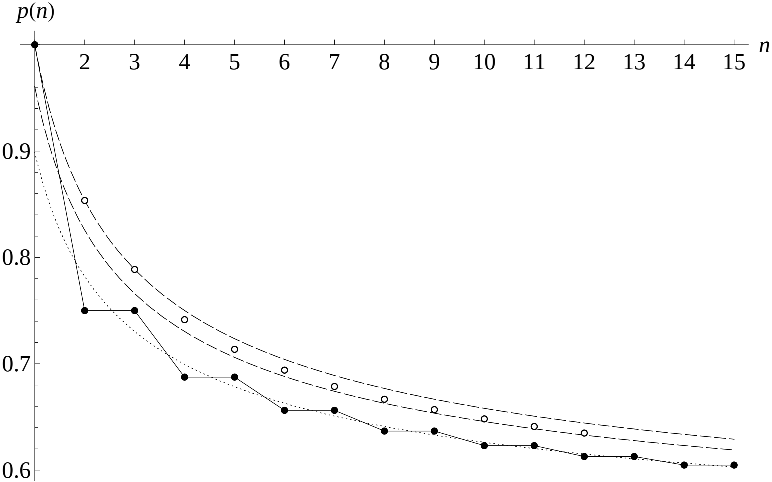
- upper bound: $p(n) \leq \dfrac{1}{2} + \dfrac{1}{2\sqrt{n}}$,
- lower bound: $p(n) \geq \dfrac{1}{2} + \sqrt{\dfrac{2}{3\pi n}}$.

Comparison of classical and quantum RACs with SR



White dots correspond to QRACs obtained using numerical optimization.
Black dots correspond to optimal classical RAC.

Open problems

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Prove the optimality of any of the numerically obtained $n \mapsto 1$ QRACs with SR for $n \geq 4$.

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- use a qudit instead of a qubit (consider also classical case),

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- allow shared entanglement?

Another open problem. . .

[Biosphere, Montreal]

Is this a QRAC?

Thank you for your attention!